středa 1. května 2024 17:23

(1) $f(x,y) = x^2y - e^{y^3x}$

$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = 2xy - y^3e^{y^3x}$   $\frac{\partial f}{\partial y \partial x}(x,y) = f_{xy}(x,y) = 2x - 3y^2e^{y^3x} - y^3 \cdot 3xy^2 e^{y^3x}$

$\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = x^2 - 3xy^2e^{y^3x}$   $\frac{\partial f}{\partial x \partial y}(x,y) = f_{yx}(x,y) = 2x - 3y^2e^{y^3x} - 3xy^2 \cdot y^3 e^{y^3x}$

totéž

(2) $$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0 & x = y = 0. \end{cases}$$

$(x,y) \neq (0,0) \to f_x(x,y) = y\frac{x^2-y^2}{x^2+y^2} + xy\frac{2x(x^2+y^2) - 2x(x^2-y^2)}{(x^2+y^2)^2} = y\frac{x^2-y^2}{x^2+y^2} + \frac{4x^2y^3}{(x^2+y^2)^2}$

$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0}\frac{0}{x} = 0$   $f_x(0,t) = t \cdot \frac{0-t^2}{0+t^2} + 0 \cdot \frac{4t^3}{(0+t^2)^2} = -t$

$(x,y) \neq (0,0) \to f_y(x,y) = x\frac{x^2-y^2}{x^2+y^2} + xy\frac{-2y(x^2+y^2) - 2y(x^2-y^2)}{(x^2+y^2)^2} = x\frac{x^2-y^2}{x^2+y^2} - \frac{4y^2x^3}{(x^2+y^2)^2}$

$f_{xy}(0,0) = \lim_{t\to 0}\frac{f_x(0,t) - f_x(0,0)}{t} = \lim_{t\to 0}\frac{-t-0}{t} = \boxed{-1}$   $f_y(t,0) = t\,\frac{t^2-0}{t^2+0} - 0 \cdot \frac{4t^3}{(t^2+0)^2} = t$

$f_{yx}(0,0) = \lim_{t\to 0}\frac{f_y(t,0) - f_y(0,0)}{t} = \lim_{t\to 0}\frac{t-0}{t} = \boxed{1}$

(3) $f(x,y) = e^{xy^2 - yx^3}$, $v = (1,-1)$, $(x,y) = (1,1)$.

$$D_v f(x,y) \overset{f \in C^1}{=} \nabla f(x,y) \cdot v = \nabla f(1,1) \cdot (1,-1) = (-2,1) \cdot (1,-1) = \boxed{-3}$$

$f_x = (y^2 - 3yx^2)e^{xy^2-yx^3}$, $f_y = (2xy - x^3)e^{xy^2-yx^3}$

$f_x(1,1) = -2$, $f_y(1,1) = 1$

Pro porovnání spočítáme z definice:

$$D_v f(x,y) = \lim_{t\to 0} \frac{f(x+tv_1, y+tv_2) - f(x,y)}{t} = \lim_{t\to 0} \frac{f(1+t, 1-t) - f(1,1)}{t}$$

$$= \lim_{t\to 0} \frac{e^{(1+t)(1-t)^2-(1-t)(1+t)^3} - 1}{t} = \lim_{t\to 0} \frac{e^{t^4+3t^3-t^2-3t} - 1}{t}$$

$(1-t^2)(1-t-1-2t-t^2)$

$=(t^2-1)\cdot t\cdot(3+t)$

$=(3t^3+t^4-3t-t^2)$

l'Hospital

$$= \lim_{t\to 0} (4t^3+9t^2-2t-3)\, e^{t^4+3t^3-t^2-3t} = \boxed{-3}$$

(4) $F(x,y) = (xy, x-y)$, $G(u,v) = (e^{uv}, u^2-v^2)$ $\quad (F, G: \mathbb{R}^2\to\mathbb{R}^2,\ G\circ F: \mathbb{R}^2\to\mathbb{R}^2)$

Pomocí věty o derivaci složeného zobrazení

$$J_F = \begin{pmatrix} (F_1)_x & (F_1)_y \\ (F_2)_x & (F_2)_y \end{pmatrix} = \begin{pmatrix} y & x \\ 1 & -1 \end{pmatrix}, \quad J_G = \begin{pmatrix} (G_1)_u & (G_1)_v \\ (G_2)_u & (G_2)_v \end{pmatrix} = \begin{pmatrix} ve^{uv} & ue^{uv} \\ 2u & -2v \end{pmatrix}$$

$$J_{G\circ F}(x,y) = \left[(J_G\circ F)\cdot J_F\right](x,y) = \begin{pmatrix} (x-y)e^{xy(x-y)} & xye^{xy(x-y)} \\ 2xy & -2(x-y) \end{pmatrix} \cdot \begin{pmatrix} y & x \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} y(x-y)e^{xy(x-y)} + xye^{xy(x-y)} & x(x-y)e^{xy(x-y)} - xye^{xy(x-y)} \\ 2xy^2 - 2(x-y) & 2x^2y + 2(x-y) \end{pmatrix}$$

$$= \begin{pmatrix} y(2x-y)e^{xy(x-y)} & x(x-2y)e^{xy(x-y)} \\ 2(xy^2-x+y) & 2(x^2y+x-y) \end{pmatrix}$$

Přímý výpočet: $G\circ F(x,y) = \left(e^{xy(x-y)}, x^2y^2-(x-y)^2\right) = \left(e^{x^2y-xy^2}, x^2y^2-x^2+2xy-y^2\right)$

$$J_{G\circ F} = \begin{pmatrix} (2xy-y^2)e^{xy(x-y)} & (x^2-2xy)e^{xy(x-y)} \\ 2xy^2-2x+2y & 2x^2y+2x-2y \end{pmatrix}$$

(5) $F(3,5)=(1,1)$, $F'(3,5)=\begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix}$, $\nabla g(1,1)=(4,4)$. Tedy

$$\nabla(g\circ F)(3,5) = g'(\underbrace{F(3,5)}_{(1,1)})\cdot F'(3,5) = (4,4)\cdot\begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix} = \boxed{(-4,4)}$$

(6)
$$f(x,y) = \begin{cases} \dfrac{x^5+y^4}{x^4+y^2} & (x,y)\neq(0,0) \\ 0 & x=y=0 \end{cases}$$

$f\in\mathcal{C}^\infty(\mathbb{R}^2\setminus\{0\})$, tedy má totální diferenciál všude na $\mathbb{R}^2\setminus\{0\}$.

Platí $f(x,0)=x$, $x\in\mathbb{R}$, tedy $f_x(0,0)=1$

$f(0,y)=y^2$, $y\in\mathbb{R}$, tedy $f_y(0,0)=0$

Má-li existovat $L=f'(0,0)=(1,0)$, musí platit $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y)-f(0,0)-L(x,y)}{\sqrt{x^2+y^2}} = 0$ (*)

Počítáme tedy limitu

$$\lim_{(x,y)\to(0,0)} \frac{\frac{x^5+y^4}{x^4+y^2} - x - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \overbrace{\frac{x^5+y^4-x^5-xy^2}{(x^4+y^2)\sqrt{x^2+y^2}}}^{g(x,y)}$$

$$g(x,x) \overset{x\neq0}{=} \frac{x^4-x^3}{|x|^3(1+x^2)} = \operatorname{sgn} x\,\frac{x-1}{x^2+1} \quad \text{(nemá limitu)}$$

Nemůže tedy platit (*) a $f$ nemá v bodě $(0,0)$ totální diferenciál.

(7)

$$f(x,y)=\begin{cases}\dfrac{x^5+y^4}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & x=y=0.\end{cases}$$

Opět $f\in\mathcal{C}^\infty(\mathbb{R}^2\setminus\{0\})$, tedy $f'$ existuje na $\mathbb{R}^2\setminus\{0\}$.

Dále $f_x(0,0)=f_y(0,0)=0$. Má-li existovat $f'(0,0)$ musí platit

$$\lim_{(x,y)\to(0,0)}\underbrace{\frac{x^5+y^4}{(x^2+y^2)^{\frac{3}{2}}}}_{g(x,y)}=0$$

$$\left|g(r\cos\alpha,r\sin\alpha)\right|\leq\frac{r^5|\cos^5\alpha|+r^4\sin^4\alpha}{r^3}=r\left(r|\cos^5\alpha|+\sin^4\alpha\right)\leq r(r+1)\overset{\text{pro } r\leq 1}{\leq}2r\xrightarrow{r\to0+}0.$$

Tedy $f'(0,0)$ existuje (je to nulový funkcionál).

(8)

$$f(x,y)=\sqrt[3]{x^3+y^3}.$$

Opět $f\in C^\infty(\mathbb{R}^2\setminus\{0\})$, tedy $f'$ existuje na $\mathbb{R}^2\setminus\{0\}$.

Dále platí
$$\left.\begin{array}{l} f(x,0)=x \ \to\ f_x(0,0)=1\\ f(0,y)=y \ \to\ f_y(0,0)=1\end{array}\right\}\Rightarrow \text{pokud } L=f'(0,0) \text{ existuje, potom } L(h_1,h_2)=h_1+h_2$$

Ovšem $f(x,x)=\sqrt[3]{2}\cdot x$ a $D_{(1,1)}f(0,0)=\sqrt[3]{2}$

Na druhou stranu, pokud $f'(0,0)$ existuje, potom $D_{(1,1)}f(0,0)=L(1,1)=1+1=2\ (\neq\sqrt[3]{2})$

Tedy $f'(0,0)$ neexistuje.